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Non-simply-connected gauge groups and rational points on elliptic curves

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Abstract: We consider the F-theory description of non-simply-connected gauge groups appearing in the $E_8 \times E_8$ heterotic string. The analysis is closely tied to the arithmetic of torsion points on an elliptic curve. The general form of the corresponding elliptic fibration is given for all finite subgroups of E_8 which are applicable in this context. We also study the closely-related question of point-like instantons on a K3 surface whose holonomy is a finite group. As an example we consider the case of the heterotic string on a K3 surface having the E_8 gauge symmetry broken to $\mathrm{SU}(9)/\mathbb{Z}_3$ or $(E_6 \times \mathrm{SU}(3))/\mathbb{Z}_3$ by point-like instantons with $\mathbb{Z}_3$ holonomy.

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Contents

1. Introduction

The subject of nonperturbative gauge symmetry has always been central to the study of string duality. The observation that the type IIA string acquires a nonperturbative gauge symmetry when compactified on a singular K3 surface is an essential ingredient in formulating a duality between this theory and the heterotic string compactified on a 4-torus [1].

The geometry of elliptic fibrations is particularly useful in analyzing the situations in which such gauge symmetries arise. This manifests itself clearly when "F-theory" compactifications are considered along the lines of [2, 3]. The general idea is that an F-theory model constructed from an elliptic K3 surface is dual to the heterotic string compactified on a 2-torus, and that F-theory constructed from an elliptic Calabi-Yau threefold is dual to the heterotic string on a K3 surface. There is a further duality in each case which arises upon further compactification on a 2-torus, in which F-theory is replaced by the type IIA string.

One of the great strengths of the F-theory (or type IIA) picture is that the geometry on the F-theory side may then be used to predict nonperturbative gauge symmetries for the heterotic string. Thus we are able to understand effects which are actually nonperturbative in both pictures at the same time.

The focus of this paper concerns the geometry of the elliptic fibrations of F-theory when the gauge symmetry in question is a non-simply-connected gauge group. This has already been studied to a small extent when the F-theory for the Spin(32)/$\mathbb{Z}_2$ heterotic string was elucidated in [4]. Here we will be more systematic and describe the general method for determining π_1 of the gauge group, at least in the case of F-theory on a K3 surface.

Once we understand how to obtain a gauge group with nontrivial π_1 it turns out to be rather easy to understand the phenomenon of "point-like instantons with discrete holonomy". This is specific to the case of the heterotic string on a K3 surface, S_H. Recall that the heterotic string requires a bundle on S_H in order to specify a particular compactification. The simplest case to understand is when this bundle corresponds to the point-like instantons of [5]. This was first applied in the context we will be discussing in [2, 3].

A point-like instanton may be loosely considered to correspond to a situation on the boundary of the moduli space of smooth bundles in which the curvature of the Yang-Mills connection is zero everywhere except at one point where it diverges. The one point is the location of the instanton. Similarly one may have multiple point-like instantons for which the curvature is concentrated at a set of points.

In the simplest case, a point-like instanton has completely trivial holonomy (with respect to a Yang-Mills connection) around any loop in S_H. (We exclude loops which pass directly through the instanton itself.) The properties of such point-like instantons have been studied at great length in [3]–[13].

We will be able to extend these methods to the case where the curvature of the instanton has been pushed into a point (or a set of points) but some discrete remnant of the holonomy remains. This is rather like the tangent bundle of a orbifold close to an orbifold point. Such instantons have also been discussed in some of the references above but mainly from the D-brane perspective (see, for example, [7]). Here we will concern ourselves with the general features of such instantons from the point of view of the elliptic fibration of F-theory. We will restrict attention to the $E_8 \times E_8$ heterotic string although there is no particular reason why the same methods should not also apply in the $\mathrm{Spin}(32)/\mathbb{Z}_2$ case.

In order to analyze the elliptic fibrations that give rise to non-simply-connected gauge groups we need to understand something about "torsion points on an elliptic curve". We review this subject in section 2. We also give all the examples that are required to study subgroups of E_8 in the context of F-theory.

In section 3 we prove the main result of the paper concerning the relationship between π_1 of the gauge symmetry of a heterotic string on a 2-torus and the Mordell-Weil group of the elliptic fibration in F-theory.

Finally in section 4 we discuss the point-like instantons with discrete holonomy for the heterotic string on a K3 surface. This is actually a very large subject and an analysis of all possibilities would be a huge undertaking. We illustrate some of the possibilities with a couple of examples.

2. Some arithmetic of elliptic curves

We shall require some basic knowledge of the arithmetic properties of elliptic curves. While most of these results are standard (see, for example, [14, 15]) we include the basic ideas here as they have not been used commonly in string theory.

We begin with everything valued in the field $\mathbb{C}$ of complex numbers for simplicity. An elliptic curve E is a smooth cubic curve in $\mathbb{P}^2$ with a marked point, $O \in E$. One may also consider an elliptic curve to be simply a 2-torus on which a point has been chosen. In the latter picture there is clearly a group action $\mathrm{U}(1)^2$ of translations of the 2-torus. This group may be identified with E itself by mapping a point P to a translation that takes O to P. Similarly if one has two points P and Q then one may define a group law by which $P+Q$ is also a point in E. Here $P+Q$ is defined by translating P by the element of $\mathrm{U}(1)^2$ associated to Q or *vice versa*.

This group law is obvious enough if E is pictured as a 2-torus but how do we picture it for a cubic curve? The answer is as follows. A generic line, $\mathbb{P}^1 \subset \mathbb{P}^2$ will intersect the cubic at three points. Thus given P and Q we may define a point R by the third point of collision of the line PQ with E. Now we may define $P+Q$ as the third point of collision of the line OR with E. It is easy to see that $O+P=P$ as required but a little more subtle to show that this group law is, in fact, associative.

In general we should take multiplicities into account when we do the addition law. That is, tangent lines should generically count as double intersections and tangent lines at a point of inflection should be counted as a triple intersection.

An important property of this group law is that it acts within an elliptic curve defined over any field k. That is, suppose we define a cubic in $\mathbb{P}^2$ where the homogeneous coordinates of $\mathbb{P}^2$ and the coefficients of the cubic lie in a field k. Then for any two points P and Q in the elliptic curve, $P+Q$ is also in the elliptic curve. For example we may take $k=\mathbb{Q}$, the rational numbers. In this case the set of rational points in the elliptic curve form a group, known as the Mordell-Weil group. Clearly this group is abelian (since "+" is commutative). The result of the celebrated Mordell–Weil theorem is that it is finitely generated.

Any elliptic curve can be mapped to $\mathbb{P}^2$ (with homogeneous coordinates $[x,y,z]$) in such a way that its equation is in "Weierstrass form":

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3\,, \tag{2.1}$$

with the chosen point O located at $[0,1,0]$. We will use affine coordinates by setting $z=1$, and regard O as the point "at infinity". (Roughly speaking, O is at $(x,y)=(0,\infty)$.) It is easily seen to be a point of inflection in E. A change of coordinates can always be used to put (2.1) in its reduced form[1]

$$y^2 = x^3 + ax + b\,. \tag{2.2}$$

We will often be interested in putting a particular point[2] at $(x,y)=(0,0)$ and we will use the more general form (2.1) with $a_6=0$.

The problem we need to solve for use later in this paper is that of writing down a general form of an elliptic curve on which a particular finite subgroup Φ of the Mordell-Weil group has been specified. For example, let us consider the case $\Phi \cong \mathbb{Z}_2$. Thus

[1] Unless k is a field of characteristic 2 or 3! But we will have no reason to consider that case here.
[2] O and $(0,0)$ are *not* the same point!

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there is a point $P \in E$, such that $P+P=O$. P is said to be a "2-torsion" point of E. Running through the above construction we see that the line passing through P twice must also pass through O. That is, the tangent line at P passes through O and so the slope, dy/dx, is infinite at P.

Putting P at $(0,0)$ implies that $a_3 = 0$. We may impose $a_1 = 0$ without loss of generality and so

$$y^2 = x(x^2 + a_2 x + a_4)\,, \tag{2.3}$$

generically has $\Phi \cong \mathbb{Z}_2$. Φ may be larger for specific values of (a_2, a_4).

We now list the possibilities we will require in this paper. In each case, we take a point P corresponding to one of the generators of our group (of maximal order n) and change coordinates to put P at $(x,y)=(0,0)$. Imposing the condition that P has order n then constrains the form of the Weierstrass equation. We also list the relationship between the coefficients of (2.2) and the discriminant $\Delta = 4a^3 + 27b^2$. See [16] for more details.

- $\Phi \cong \mathbb{Z}_2$:

$$y^2 = x(x^2 + a_2 x + a_4)\,, \tag{2.4}$$

and

$$\begin{aligned} a &= a_4 - \tfrac{1}{3}a_2^2 \\ b &= \tfrac{1}{27}a_2(2a_2^2 - 9a_4) \\ \Delta &= a_4^2(4a_4 - a_2^2) \end{aligned} \tag{2.5}$$

- $\Phi \cong \mathbb{Z}_3$:

$$y^2 + a_1 xy + a_3 y = x^3 \tag{2.6}$$

and

$$\begin{aligned} a &= \tfrac{1}{2}a_1a_3 - \tfrac{1}{48}a_1^4 \\ b &= \tfrac{1}{4}a_3^2 + \tfrac{1}{864}a_1^6 - \tfrac{1}{24}a_1^3a_3 \\ \Delta &= \tfrac{1}{16}a_3^3(27a_3 - a_1^3)\,. \end{aligned} \tag{2.7}$$

- $\Phi \cong \mathbb{Z}_4$:

$$y^2 + a_1 xy + a_1 a_2 y = x^3 + a_2 x^3 \tag{2.8}$$

and

$$\begin{aligned} a &= -\tfrac{1}{48}a_1^4 + \tfrac{1}{3}a_1^2a_2 - \tfrac{1}{3}a_2^2 \\ b &= \tfrac{1}{864}(a_1^2 - 8a_2)(a_1^4 - 16a_1^2a_1 - 8a_2^2) \\ \Delta &= -\tfrac{1}{16}a_1^2a_2^4(a_1^2 - 16a_2)\,. \end{aligned} \tag{2.9}$$

- $\Phi \cong \mathbb{Z}_5$:

$$y^2 + a_1 xy + (a_1 - b_1)b_1^2\, y = x^3 + (a_1 - b_1)b_1\, x^2 \tag{2.10}$$

and

$$\begin{aligned} a &= \tfrac{1}{6}a_1 b_1^3 - \tfrac{1}{48}a_1^4 + \tfrac{1}{3}a_1^2 b_1^2 - \tfrac{1}{3}b_1^4 - \tfrac{1}{6}a_1^3 b_1 \\ b &= \tfrac{1}{864}(a_1^2 - 2a_1 b_1 + 2b_1^2)(a_1^4 + 14a_1^3 b_1 + 26a_1^2 b_1^2 - 116 a_1 b_1^3 + 76 b_1^4) \\ \Delta &= \tfrac{1}{16}(a_1^2 + 9\, a_1 b_1 - 11 b_1^2)(a_1 - b_1)^5 b_1^5 \,. \end{aligned} \tag{2.11}$$

- $\Phi \cong \mathbb{Z}_6$:

$$y^2 + a_1 xy + \tfrac{1}{32}(a_1 - b_1)(3a_1 + b_1)(a_1 + b_1) = x^3 + \tfrac{1}{8}(a_1 - b_1)(a_1 + b_1)x^2 \tag{2.12}$$

and

$$\begin{aligned} a &= \tfrac{1}{192} b_1(3a_1^3 - 3a_1^2 b_1 - 3a_1 b_1^2 - b_1^3) \\ b &= \tfrac{1}{110592}(3a_1^2 - 6a_1 b_1 - b_1^2)(9a_1^4 - 6a_1^2 b_1^2 - 24 a_1 b_1^3 - 11 b_1^4) \\ \Delta &= \tfrac{1}{2^{24}}(a_1 - 5b_1)(3a_1 + b_1)^2(a_1 + b_1)^3(a_1 - b_1)^6 \,. \end{aligned} \tag{2.13}$$

- $\Phi \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$:

$$y^2 = x(x - b_2)(x - c_2) \tag{2.14}$$

and

$$\begin{aligned} a &= \tfrac{1}{3}(b_2 c_2 - b_2^2 - c_2^2) \\ b &= -\tfrac{1}{27}(b_2 + c_2)(b_2 - 2c_2)(2b_2 - c_2) \\ \Delta &= -b_2^2 c_2^2 (b_2 - c_2)^2 \,. \end{aligned} \tag{2.15}$$

- $\Phi \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$:

$$y^2 + a_1 xy - a_1(b_1^2 - \tfrac{1}{16}a_1^2)y = x^3 - (b_1^2 - \tfrac{1}{16}a_1^2)x^2 \tag{2.16}$$

and

$$\begin{aligned} a &= -\tfrac{1}{768}a_1^4 - \tfrac{7}{24}a_1^2 b_1^2 - \tfrac{1}{3}b_1^4 \\ b &= \tfrac{1}{55296}(a_1^2 + 16b_1^2)(a_1^2 - 24a_1 b_1 + 16b_1^2)(a_1^2 + 24 a_1 b_1 + 16 b_1^2) \\ \Delta &= -\frac{1}{2^{16}} a_1^2 b_1^2 (a_1 - 4b_1)^4 (a_1 + 4b_1)^4 \,. \end{aligned} \tag{2.17}$$

The subscripts for the coefficient parameters in the above have been chosen to correspond to their degree under a rescaling symmetry $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$. In each case there are two parameters leading to an effective one parameter family when this rescaling is taken into account.

The power of having formulated the above discussion over an arbitrary field becomes apparent when we choose the field to be $k = \mathbb{C}(s)$, the field of functions of the form $p(s)/q(s)$ where $p(s)$ and $q(s)$ are polynomials in some new variable s. This corresponds to a family of elliptic curves $\pi : \mathscr{X} \to D$, where the generic fibre is an elliptic curve and D is a complex line with coordinate s.[3] A "rational" point in the field $\mathbb{C}(s)$ is now a *section* of the bundle $\pi : \mathscr{X} \to D$.

Thus we may define the Mordell-Weil group of a family of elliptic curves as the group of sections. The group law is exactly that as explained above.

Note that rational points on an elliptic curve over $\mathbb{Q}$ can be trivially rewritten as rational points on an elliptic curve over $\mathbb{C}(s)$. The converse is not true however. Mazur's theorem [17, 18] asserts that only fifteen possibilities are allowed for the torsion part of Φ for an elliptic curve over $\mathbb{Q}$. One possibility which is not allowed is $\Phi \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$. This *is* allowed for an elliptic curve over $\mathbb{C}(s)$ however. In this case one may put

- $\Phi \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$:

$$\begin{aligned} y^2 + a_1xy - \tfrac{1}{3}(a_1+\omega b_1)(a_1+\omega^2 b_1)b_1\, y \\ = x^3 - (a_1-b_1)b_1\, x^2 + \tfrac{1}{3}(a_1+\omega b_1)(a_1+\omega^2 b_1)b_1^2\, x \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} a &= -\tfrac{1}{48}a_1(a_1-2b_1)(a_1-2\omega b_1)(a_1-2\omega^2 b_1) \\ b &= \tfrac{1}{864}(a_1^2+2a_1b_1-2b_1^2)(a_1^2+2\omega a_1b_1-2\omega^2b_1^2)(a_1^2+2\omega^2a_1b_1-2\omega b_1^2) \\ \Delta &= \tfrac{1}{432}(a_1+b_1)^3(a_1+\omega b_1)^3(a_1+\omega^2 b_1)^3 b_1^3\,, \end{aligned} \tag{2.19}$$

where ω is a nontrivial cube cube of unity. Here one of the $\mathbb{Z}_3$ generators may be put at $(0,0)$ but any generator of the other $\mathbb{Z}_3$, e.g., $(\frac{1}{3}(a_1-2b_1)b_1 - \frac{\omega}{3}(a_1+b_1)b_1, 0)$, fails to lie in $\mathbb{Q}$.

3. The heterotic string on a two-torus

We begin with a quick statement of the elements of F-theory we require. For a more thorough description see [2, 3, 19, 20, 11] and in particular [21]. We will treat the duality between F-theory modeled on a space Y and the heterotic string compactified on a space Z as a limit of the duality between the type IIA string compactified on Y and the heterotic string compactified on $Z \times T^2$.

In this section we consider F-theory on a K3 surface, S_F. This is dual to the $E_8 \times E_8$ heterotic string on a 2-torus. Of particular interest is the case where this 2-torus is very large. In this case S_F degenerates into a reducible variety consisting of two rational elliptic surfaces, R_1 and R_2, intersecting along an elliptic curve, E_*. The complex

[3] More generally, we could consider the field of rational functions on an arbitrary base manifold D (if it has an algebraic structure), such as a Riemann surface or a higher-dimensional algebraic variety.

structure of E_* is identified with that of the heterotic 2-torus [20]. A similar picture for the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string was constructed in [11]. (See also [22].)

The geometry of these rational elliptic surfaces is of central importance to us. Roughly speaking, each of R_1 and R_2 is identified with one of the E_8 gauge groups of the heterotic string. In particular, holding E_* fixed, deforming the complex structure on R_i is dual to deforming the corresponding E_8 bundle on T^2.

Let the proper structure group of the E_8-bundle on T^2 be $\mathscr{G} \subset E_8$. The observed gauge group in the heterotic string theory from this primordial E_8 gauge symmetry will then be the centralizer of $\mathscr{G} \subset E_8$.

Dual to this, on the F-theory side, the gauge symmetry is produced by 2-spheres shrinking down to zero area within R_i. Let us consider this from the elliptic fibration $\pi : R_i \to f$, where f is a $\mathbb{P}^1$ and the generic fibre is an elliptic curve. This fibration has at least one section which we label σ_0. The elements of $H_2(R_i, \mathbb{Z})$ corresponding to σ_0 and the elliptic fibre form a 2-dimensional unimodular even sublattice, $U \subset H_2(R_i, \mathbb{Z})$. The fact that $H_2(R_i, \mathbb{Z})$ is itself even and unimodular of rank 10 shows that $H_2(R_i, \mathbb{Z}) \cong \Gamma_8 \oplus U$, where Γ_8 is isomorphic to the root lattice of E_8.[4] In order to analyze how 2-spheres can shrink down we need the following [23]

Theorem 1 *Let M be the lattice of 2-cycles within the fibres of the elliptic fibration $\pi : R_i \to f$ which do* not *intersect σ_0. Then*

$$0 \to M \to \Gamma_8 \to \Phi \to 0\,, \tag{3.1}$$

where Φ is the Mordell-Weil group of this elliptic fibration.

The F-theory rule for analyzing the gauge group is that all components in all the fibres of $\pi : R_i \to f$ should be shrunk to zero size. The subset of elements of Γ_8, of self-intersection -2, which are shrunk to zero size in this process form the roots of the observed gauge symmetry *algebra*. This subset is given precisely by the generators of M. That is, M is the root lattice of the gauge algebra.

In the generic case, all of the fibres of $\pi : R_i \to f$ are fibres of Kodaira type I_1 and thus M is trivial. In this case Φ is of rank 8 and generates all of Γ_8. No elements of $H_2(R_i, \mathbb{Z})$ corresponding to roots of Γ_8 are shrunk to zero size when the fibres are shrunk and so the gauge algebra is trivial. The structure group of the bundle really is the full E_8.

At the other extreme, if one acquires a type II^* fibre then $M \cong \Gamma_8$ and Φ becomes trivial. The gauge algebra is $\mathfrak{e}_8$ and the bundle has trivial holonomy. These are point-like instanton solutions [2].

We would like to know the gauge *group*, $\mathscr{G}$, as well as the gauge algebra for the F-theory picture. We will do this by generalizing a method used for analysis of the

[4]The intersection form in $H_2(R_i, \mathbb{Z})$ gives Γ_8 a negative signature. In our discussion of Lie algebras we will implicitly assume this signature was positive. Although this is sloppy it helps simplify the discussion.

$\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string in [4]. We require more than just a knowledge of the massless vector states which give the gauge bosons. We also need to analyze the representations of massive states with respect to $\mathscr{G}$.

To do this consider the BPS solitons of the type IIA string. These correspond to 2-branes wrapped over elements of $H_2(R_i, \mathbb{Z})$. The Ramond-Ramond charges of these states are given by this homology class. This allows one to determine exactly the representation theory for the BPS states when we have an enhanced gauge symmetry.

Given the exact sequence (3.1) we may use standard methods in algebra (see, for example, chapter 3 of [24]) to yield

$$0 \to \mathrm{Hom}(\Phi, \mathbb{Z}) \to \mathrm{Hom}(\Gamma_8, \mathbb{Z}) \xrightarrow{j} \mathrm{Hom}(M, \mathbb{Z}) \to \mathrm{Ext}(\Phi, \mathbb{Z}) \to 0 \,. \tag{3.2}$$

Now Φ may be decomposed into its free part $\mathrm{Free}(\Phi)$ and its torsion part $\mathrm{Tors}(\Phi)$. We have $\mathrm{Hom}(\Phi, \mathbb{Z}) \cong \mathrm{Free}(\Phi)$ and $\mathrm{Ext}(\Phi, \mathbb{Z}) \cong \mathrm{Tors}(\Phi)$. Also $\mathrm{Hom}(\Gamma_8, \mathbb{Z}) \cong \Gamma_8$ and $\mathrm{Hom}(M, \mathbb{Z})$ is the *weight lattice* of the algebra of $\mathscr{G}$.

Finite group	Centralizer	Kodaira Fibres
$\{\mathrm{id}\}$	E_8	II^*, $2\mathrm{I}_1$ *or* II^*, II
$\mathbb{Z}_2$	$\mathrm{Spin}(16)/\mathbb{Z}_2$	I_4^*, $2\mathrm{I}_1$
$\mathbb{Z}_2$	$(E_7 \times \mathrm{SU}(2))/\mathbb{Z}_2$	III^*, I_2, I_1 *or* III^*, III
$\mathbb{Z}_3$	$\mathrm{SU}(9)/\mathbb{Z}_3$	I_9, $3\mathrm{I}_1$
$\mathbb{Z}_3$	$(E_6 \times \mathrm{SU}(3))/\mathbb{Z}_3$	IV^*, I_3, I_1 *or* IV^*, IV
$\mathbb{Z}_4$	$(\mathrm{SU}(8) \times \mathrm{SU}(2))/\mathbb{Z}_4$	I_8, I_2, $2\mathrm{I}_1$
$\mathbb{Z}_4$	$(\mathrm{Spin}(10) \times \mathrm{Spin}(6))\mathbb{Z}_4$	I_1^*, I_4, I_1
$\mathbb{Z}_5$	$(\mathrm{SU}(5) \times \mathrm{SU}(5))/\mathbb{Z}_5$	$2\mathrm{I}_5$, $2\mathrm{I}_1$
$\mathbb{Z}_6$	$(\mathrm{SU}(6) \times \mathrm{SU}(3) \times \mathrm{SU}(2))/\mathbb{Z}_6$	I_6, I_3, I_2, I_1
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(\mathrm{Spin}(12) \times \mathrm{Spin}(4))/(\mathbb{Z}_2 \times \mathbb{Z}_2)$	I_2^*, $2\mathrm{I}_2$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$(\mathrm{Spin}(8) \times \mathrm{Spin}(8))/(\mathbb{Z}_2 \times \mathbb{Z}_2)$	$2\mathrm{I}_0^*$
$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$(\mathrm{SU}(4)^2 \times \mathrm{SU}(2)^2)/(\mathbb{Z}_4 \times \mathbb{Z}_2)$	$2\mathrm{I}_4$, $2\mathrm{I}_2$
$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	$\mathrm{SU}(3)^4/(\mathbb{Z}_3 \times \mathbb{Z}_3)$	$4\mathrm{I}_3$

Table 1: Finite abelian subgroups of E_8 with at most two generators, their centralizers, and the Kodaira fibres of the corresponding rational elliptic surface.

Of particular interest is the map j in (3.2). This is mapping the RR-charges of the BPS states into the weight lattice of the gauge algebra. If the gauge group is not simply connected we expect certain representations of the algebra to be missing. This is measured by the cokernel of j which is isomorphic to $\mathrm{Ext}(\Phi, \mathbb{Z}) \cong \mathrm{Tors}(\Phi)$. In fact, a little group theory yields the following

Theorem 2 *If the BPS solitons form a faithful representation of the gauge group $\mathscr{G}$, then*

$$\pi_1(\mathscr{G}) \cong \mathrm{Tors}(\Phi) \,. \tag{3.3}$$

There is a further check that could be made of our claim that $\pi_1(\mathscr{G})$ is non-trivial — we could analyze the behavior of the theory on non-trivial spacetimes. We will not attempt to do this in detail, but just point out that when the theory is further compactified on T^2, we would expect to see $\pi_1(\mathscr{G})$ as a symmetry of the theory. And indeed, as a subgroup of the Mordell-Weil group it acts on the compactified IIA string.

Fortunately all possible forms of the rational elliptic surface as an elliptic fibration have been listed by Persson [25] and Φ has been determined in each case. In particular, the list of possible torsion groups is completely known. It appears in the first column of table 1.

Each of these torsion groups can be embedded in E_8, with one or two inequivalent embeddings in each case. (The equivalence is conjugacy within E_8.) In fact, this is the complete list of conjugacy classes of finite abelian subgroups of E_8 with at most two generators.[5] If we calculate the centralizer of each embedded torsion group, we will determine the maximum gauge group $\mathscr{G}$ for each possible π_1. This is done in the second column of the table. All of the corresponding groups have rank 8. And in fact, each of these possibilities is realized in F-theory, due to the classification of rational elliptic surfaces of this type given in [23]. We list the Kodaira fibres of the corresponding rational surfaces[6] in the third column of table 1.

For example, consider one of the rational elliptic surfaces listed in [23, 25] which consists of a fibration with Kodaira fibres of the class one of I_9 and three of I_1. According to the usual rules of F-theory, the gauge algebra associated to the I_9 fibre will be $\mathfrak{su}(9)$. Persson tells us that $\Phi \cong \mathbb{Z}_3$ in this case however and so $\mathscr{G} \cong \mathrm{SU}(9)/\mathbb{Z}_3$.

Note that $\mathrm{SU}(9)/\mathbb{Z}_3$ is a centralizer of $\mathbb{Z}_3 \subset E_8$ and so this corresponds to a heterotic string on a bundle with holonomy $\mathbb{Z}_3$. That is, one of the Wilson loops of the heterotic 2-torus is such that going around it three times is trivial.

Note that in each case $\pi_1(\mathscr{G})$ is actually isomorphic to the discrete centralizer. This is a special property of E_8, due to the E_8 lattice being unimodular.

4. The heterotic string on a K3 surface

Statements about duality become more interesting when we consider F-theory on a Calabi-Yau threefold, X, and the dual picture of a $E_8 \times E_8$ heterotic string on a K3 surface, S_H. Since K3 is simply-connected we will need point-like instantons to generate the finite holonomy of the preceding section.

To do the analysis we may write S_H as an elliptic fibration $\pi_H : S_H \to B$, where $B \cong \mathbb{P}^1$ and apply the duality of the preceding section "fibre-wise". We will assume X may be written in the form of a K3 fibration $p : X \to B$ and an elliptic fibration

[5] More than two generators would not be appropriate on the heterotic side, since we only have two Wilson lines available to generate the holonomy.

[6] In a few cases, two surfaces are possible — the one with the I_k fibres is generic.

$\pi_F : X \to \Sigma$ with at least one section, σ_0. See [19] for a description of the conditions required for this assumption.

As S_H becomes large, X undergoes a stable degeneration in which each generic K3 fibre of the map p, becomes a union $R_1 \cup_{E_*} R_2$ of two rational elliptic surfaces intersecting along an elliptic curve just as in the last section. S_H may then be identified as the elliptic fibration $\pi_H : S_H \to B$ with fibre isomorphic to E_*.

Again gauge symmetry is produced by 2-cycles shrinking down within a generic R_i when the elliptic fibre of the fibration $\pi_F : X \to \Sigma$ is shrunk down to zero size. Globally this corresponds to 4-cycles collapsing onto 2-cycles within X.

This is most easily represented by thinking of the discriminant locus, Δ, of the fibration $\pi_F : X \to \Sigma$. Δ corresponds to a divisor within Σ. The degeneration of the fibre of Kodaira type worse than I_1 or II along a component of Δ will give rise to an enhanced gauge symmetry. See [2, 3, 19] for more details.

Σ may also be written as a $\mathbb{P}^1$-fibration over B. Let f denote a generic $\mathbb{P}^1$ fibre of this fibration. Thus R_i is an elliptic fibration of f as in the previous section. We may restrict the fibration $\pi_F : X \to B$ to each f and look at the Mordell-Weil group of each rational elliptic surface as in the previous section. Again, any torsion here will cause an obstruction to certain BPS states forming any representation of the gauge algebra.

A new aspect arises over singularities of Δ such as when irreducible components of Δ collide. The fibre over such a singularity (which need not be within Kodaira's classification) may contain 2-cycles whose homology class did not appear in nearby fibres. The result of wrapping 2-branes over such cycles results in hypermultiplets. Thus, in order to find which representations of the gauge group appear we should extend the analysis of the previous section to such singularities within Δ. The analysis of the Mordell-Weil group is much more difficult in this case (due to possibilities such as "birational sections" which were not an issue in lower dimension), but it seems likely that a result similar to the conclusion of theorem 2 still holds, namely, that $\pi_1(\mathscr{G}) \cong \mathrm{Tors}(\Phi)$ where Φ is the Mordell-Weil group. We will restrict our attention to models for which this does hold.

The possibilities one may analyze are extremely numerous and there are many interesting features about most of the possibilities. Here we will just give an example to give the flavour of the subject. The Weierstrass forms which would be necessary to compute any example are listed in section 2.

The method we use is standard to F-theory and comes from [3]. The notation we use follows [4, 11, 21]. In particular we take Σ to be the Hirzebruch surface $\mathbb{F}_n$ and C_0 to be a $\mathbb{P}^1$ section of Σ fibred over B with self-intersection $-n$. In the heterotic string picture this describes an E_8-bundle with $c_2 = 12 - n$. The methods used to analyze the geometry are very similar to those appearing in the above references and for that reason we focus mainly on the results rather than details of the method.

4.1. $\mathbb{Z}_3$ Point-like instantons

Suppose we have a point-like instanton on a K3 surface whose holonomy is $\mathbb{Z}_3$. This only makes sense if the topology of the boundary of a small neighbourhood around the instanton has $\pi_1 \supset \mathbb{Z}_3$. One way to arrange this is to have the instanton located on a $\mathbb{C}^2/\mathbb{Z}_3$ singularity of the K3 surface. In this case the boundary of a small neighbourhood around the instanton is a lens space $S^3/\mathbb{Z}_3$.

We will just focus on one of the E_8's of the heterotic string. We expect E_8 to be broken to the centralizer of $\mathbb{Z}_3 \subset E_8$. There are two possibilities for embedding $\mathbb{Z}_3$ in E_8 and thus two possible centralizers. Each has quite different geometry and thus physics.

One possibility is $\mathrm{SU}(9)/\mathbb{Z}_3$. We know that because of the $\mathbb{Z}_3$, the Weierstrass form must be given by (2.6). We may achieve the curve of I_9 fibres along C_0 required to generate the $\mathfrak{su}(9)$ symmetry by making a_3 vanish to order 3 along this curve. The resulting discriminant is then forced into the form shown in figure 1.

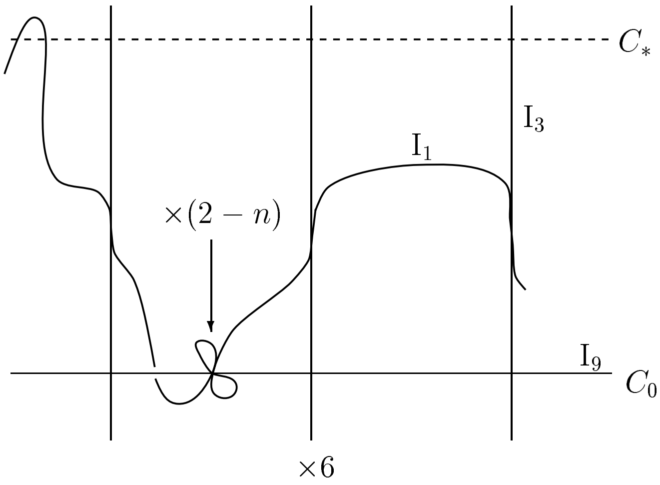

Figure 1: A perturbative gauge group of $\mathrm{SU}(9)/\mathbb{Z}_3$.

The first thing to note is that six lines of I_3 fibres appear in the f direction. Each such line crosses C_*. Recall that C_* is the section of S_H as an elliptic fibration. This implies that S_H *generically has six quotient singularities of the form* $\mathbb{C}^2/\mathbb{Z}_3$.

Thus the F-theory picture is in nice agreement with the heterotic picture. In order to get an $\mathrm{SU}(9)/\mathbb{Z}_3$ gauge symmetry we must have fractionally charged instantons and thus S_H must have quotient singularities. The key to this happening in the F-theory picture is that the form of the discriminant 2.7) has a factor which is a perfect cube. Note that all of the forms of the discriminant listed in section 2 have strong factorization properties and so this is a feature which is common to all examples. Actually F-theory also managed to correctly count the number of orbifold points in S_H as we now argue.

The bundle, $V \to S_H$ for the heterotic string has holonomy $\mathbb{Z}_3$. This implies that there is a map $\phi : S_C \to S_H$ which is three-to-one such that $\phi^* V$ has trivial holonomy. ϕ will have fixed points at the location of the orbifold points. Let the number of orbifold points be m. Given that S_C is a K3 surface[7] and that S_H is a K3 surface, one may do

[7] One might also think that S_C could be a 4-torus. However, in that case S_H would not be the Weierstrass model of an elliptic surface with a section.

a simple Euler characteristic calculation to determine m:

$$\frac{24-m}{3} + 3m = 24\,. \tag{4.1}$$

Thus $m = 6$ in agreement with F-theory.

As well as producing a $\mathbb{C}^2/\mathbb{Z}_3$ orbifold point in S_H, each vertical line of I_3 fibres also produces a nonperturbative gauge symmetry of $\mathfrak{su}(3)$. Thus, a $\mathbb{C}^2/\mathbb{Z}_3$ orbifold point with this fractional point-like instanton produces a nonperturbative gauge enhancement of $\mathfrak{su}(3)$. The collisions of the vertical lines of I_3 fibres with the line of I_9 fibres along C_0 each produce hypermultiplets in the $(\mathbf{3},\mathbf{9})$ representation of $\mathfrak{su}(3)\oplus\mathfrak{su}(9)$. In resolving this collision there is a small resolution which must be done. This involves moving off in a Kähler cone direction whose existence implies a contribution to $H^2(X)$ from the *free* part of Φ. This produces an extra U(1) gauge symmetry [3]. Assuming this happens independently for each of the six I_9-I_3 collisions we end up with a gauge group

$$\mathscr{G} \cong \frac{\mathrm{S}\big(\mathrm{U}(9)\times\mathrm{U}(3)^6\big)}{\mathbb{Z}_3}\,. \tag{4.2}$$

The remaining I_1 part of the discriminant collides with degree 3 with the lines of I_3 fibres and I_9 fibres. The collisions with I_3 lines produce nothing interesting but the collisions with C_0 at a total of $2-n$ points each produce a massless tensor. Using arguments along the lines of [21] one may analyze the moduli space of the object associated with each tensor. One finds that it corresponds to a copy of S_H. That is, it is a free object allowed to be anywhere in S_H. In fact, these massless tensors are essentially identical to the usual point-like E_8 instantons with trivial holonomy of [6, 2]. Such point-like instantons are known to have $c_2 = 1$. Since the total c_2 of this bundle should be $12-n$, the fractional point-like instantons appear to each contribute $((12-n)-(2-n))/6 = \frac{5}{3}$ towards c_2.

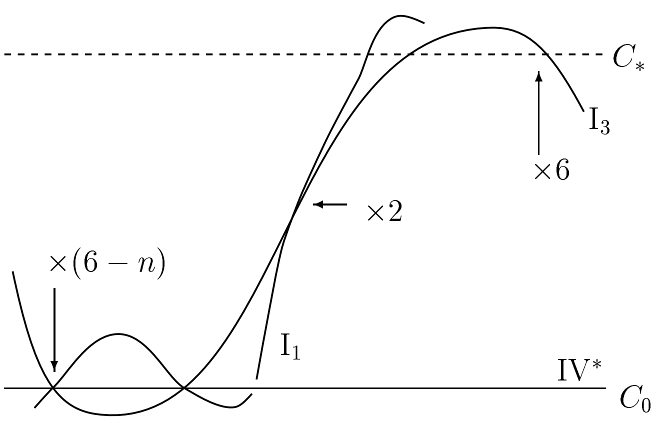

Figure 2: A perturbative gauge group of $(E_6\times\mathrm{SU}(3))/\mathbb{Z}_3$.

As usual, one may check the anomalies of this six-dimensional theory (see, for example, [26]). When doing this it is important to remember that by going to the stable degeneration we are only looking at "half" of the $E_8\times E_8$ heterotic string. The anomalies will only cancel if the *full* massless particle spectrum is determined.

Let us now consider another example. $\mathbb{Z}_3$ may be embedded in E_8 in an inequivalent way such that it centralizes $(E_6\times\mathrm{SU}(3))/\mathbb{Z}_3$. We may put a line of type IV^* fibres along C_0 to achieve this. The result is shown in figure 2.

The curve of I_3 fibres crosses C_* six times. Thus the heterotic K3 surface, S_H, is forced to have six $\mathbb{C}^2/\mathbb{Z}_3$ orbifold points again. Beyond this however, this situation seems altogether milder than the $\mathrm{SU}(9)/\mathbb{Z}_3$ case above. There are no new nonperturbative gauge groups. There are $(6-n)$ blow-ups in the base implying $6-n$ old-fashioned point-like E_8 instantons. Therefore each fractional instanton contributes $c_2 = 1$ to get a total $c_2 = 12 - n$.

Note that we can obtain more possibilities from these two $\mathbb{Z}_3$ examples. One may allow some of the point-like instantons with trivial holonomy to coalesce with the orbifold points containing the fractional point-like instantons. This yields many more nonperturbative gauge groups, tensors and hypermultiplets. The situation is very similar to that described in [11] so we will not pursue it further here.

Note added

When a specific torsion subgroup Φ is imposed on all elliptic curves in a family, as happened in the situations studied in this paper, the monodromy group of the family of elliptic curves is reduced from $SL(2,\mathbb{Z})$ to a subgroup Γ of finite index. In the case $\Phi = \mathbb{Z}_n$, the corresponding monodromy group is known as $\Gamma_1(n)$, and in the case $\Phi = \mathbb{Z}_n \times \mathbb{Z}_n$, the corresponding monodromy group is known as $\Gamma(n)$. There is a closely related monodromy group $\Gamma_0(n)$ corresponding to a cyclic subgroup in which a generator has not been chosen.

As this paper was being put into final form, two interesting papers appeared[27, 28] which study F-theory models with reduced monodromy $\Gamma_0(n)$ and $\Gamma(n)$. The analysis given here concerns conventional F-theory compactifications using such families of elliptic curves, whereas [27, 28] consider unconventional compactifications in which an additional field has been turned on.

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